"Solutions of a tracer transport problem with a variable vertical eddy diffusivity"

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Eric Deleersnijder, March 6, 2014

The problem to solve

Let t denote the time. If x and y denote the horizontal coordinates while z is the vertical coordinate (increasing upward), the domain of interest is defined by the inequalities

$$-\infty < x < \infty \ , \quad -\infty < y < \infty \ , \quad -h < z < 0 \ , \tag{1}$$

where the constant h is the depth of the sea. As will be seen, it is convenient to introduce the normalised vertical coordinate

$$\sigma = \frac{z+h}{h} \ , \tag{2}$$

with $0 < \sigma < 1$. The seabed is located at $\sigma = 0$, whereas $\sigma = 1$ is the equation of the sea surface (Figure 1). The aforementioned horizontal boundaries are assumed to be impermeable.

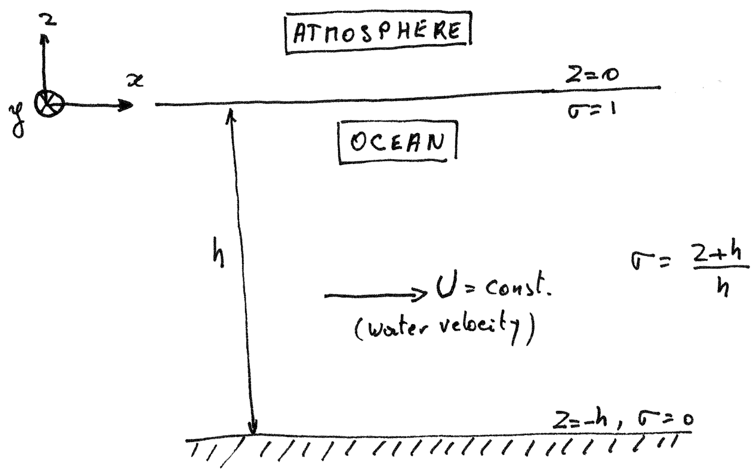

Figure 1. Geometry of the domain of interest. The horizontal coordinate are denoted x and y, while z is the vertical coordinate, increasing upward. The domain of interest is infinite in the horizontal direction ($-\infty < x < \infty$, $-\infty < y < \infty$), while its height is finite ($-h < z < 0$). The water is flowing in the direction of the x-axis with the constant velocity U. For the sake of simplicity, the normalised vertical coordinate $\sigma = (z+h)/h$ is used in most of the calculations.

The water velocity, U, is constant, horizontal and parallel to the x-axis. It is reasonable to assume that the horizontal diffusivity, K_h, is constant. The vertical diffusivity is not necessarily taken to be a constant, thereby allowing one to take into account the impact of the bottom and surface boundaries. Accordingly, the vertical diffusivity is denoted

$$K_v(\sigma) \;=\; \bar{K}_v\,\kappa(\sigma) \tag{3}$$

where the constant $\bar{K}_v$ is the depth-averaged diffusivity, i.e.

$$\bar{K}_v \;=\; \int_0^1 K_v(\sigma)\,d\sigma \;, \tag{4}$$

implying that

$$\int_0^1 \kappa(\sigma)\,d\sigma \;=\; 1 \tag{5}$$

Now consider a tracer undergoing a first-order decay process (e.g. radioactivity or mortality) characterised by the decay rate γ, where the latter is a non-negative constant. The tracer under study is also injected into the domain at rate $q(t,x,y,z)$. If the constant ρ represents the density of the water (Boussinesq approximation), this source function is such that the mass of tracer injected into the volume element $[x,x+\delta x]\times[y,y+\delta y]\times[z,z+\delta z]$ during the time interval $[t,t+\delta t]$ tends to $\rho\,q(t,x,y,z)\delta x\,\delta y\,\delta z\,\delta t$ as $\delta x,\delta y,\delta z \to 0$ and $\delta t \to 0$; the dimension of the source function q is time^{-1}.

Let the mass fraction $C(t,x,y,z)$ represent the concentration of the tracer under study. The latter obeys the reactive-transport equation

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} \;=\; q \;-\; \gamma C \;+\; K_h\frac{\partial^2 C}{\partial x^2} \;+\; K_h\frac{\partial^2 C}{\partial y^2} \;+\; \frac{\partial}{\partial z}\left(K_v\frac{\partial C}{\partial y}\right) . \tag{6}$$

Combining (2), (3) and (6) yields

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} \;=\; q \;-\; \gamma C \;+\; K_h\frac{\partial^2 C}{\partial x^2} \;+\; K_h\frac{\partial^2 C}{\partial y^2} \;+\; \frac{\bar{K}_v}{h^2}\frac{\partial}{\partial \sigma}\left(\kappa\frac{\partial C}{\partial \sigma}\right) , \tag{7}$$

which is to be solved under the initial condition

$$C(0,x,y,\sigma) \;=\; C^0(x,y,\sigma) \tag{8}$$

and the bottom and surface impermeability conditions

$$\left[\kappa\frac{\partial C}{\partial \sigma}\right]_{\sigma=0} \;=\; 0 \;=\; \left[\kappa\frac{\partial C}{\partial \sigma}\right]_{\sigma=1} . \tag{9}$$

Generic solution

If G denotes the Green's function of the partial differential problem (7)-(9), the general expression of its solution is

$$\begin{aligned} C(t,x,y,\sigma) \;=\; & \overbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^1 C^0(x',y',\sigma')\,e^{-\gamma t}\,G(t,x-x',y-y',\sigma,\sigma')\,d\sigma'\,dy'\,dx'}^{\text{response to the initial condition}} \\ & + \underbrace{\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^1 q(t',x',y',\sigma')\,e^{-\gamma(t-t')}\,G(t-t',x-x',y-y',\sigma,\sigma')\,d\sigma'\,dy'\,dx'\,dt'}_{\text{response to the source function}} \end{aligned} \tag{10}$$

Let the function δ represent the Dirac impulse. Then, the Green's function is the solution of the following transport problem

$$\frac{\partial G}{\partial t} + U\frac{\partial G}{\partial x} = \delta(t-0)\delta(x-0)\delta(y-0)\delta(\sigma-\sigma') + K_h\frac{\partial^2 G}{\partial x^2} + K_h\frac{\partial^2 G}{\partial y^2} + \frac{\bar{K}_v}{h^2}\frac{\partial}{\partial \sigma}\left(\kappa\frac{\partial G}{\partial \sigma}\right) \tag{11}$$

$$G(0^-,x,y,\sigma,\sigma') = 0 \tag{12}$$

$$\left[\kappa\frac{\partial G}{\partial \sigma}\right]_{\sigma=0} = 0 = \left[\kappa\frac{\partial G}{\partial \sigma}\right]_{\sigma=1} \tag{13}$$

In the domain of interest defined by (1), the solution to (11)-(13) reads

$$G(t,x,y,\sigma,\sigma') = \frac{\exp\left[-\frac{(x-Ut)^2}{4K_h t}\right]}{\sqrt{4\pi K_h t}}\;\frac{\exp\left[-\frac{y^2}{4K_h t}\right]}{\sqrt{4\pi K_h t}}\;\sum_{n=0}^{\infty}\exp\left(-\frac{\bar{K}_v\lambda_n t}{h^2}\right)\psi_n(\sigma')\,\psi_n(\sigma) \tag{14}$$

where λ_n and ψ_n are the eigenvalues and eigenfunctions, respectively, of the vertical diffusion operator, which will be dealt with in the next Section.

Eigenmodes of the vertical diffusion operator

The eigenvalues and eigenfunctions of the vertical diffusion operator arise from the following Sturm-Liouville problem

$$\frac{d}{d\sigma}\left[\kappa(\sigma)\frac{d\psi_n}{d\sigma}\right] = -\lambda_n\psi_n \ , \tag{15}$$

$$\left[\kappa(\sigma)\frac{d\psi_n}{d\sigma}\right]_{\sigma=0} = 0 = \left[\kappa(\sigma)\frac{d\psi_n}{d\sigma}\right]_{\sigma=1} \tag{16}$$

with $n = 0,1,2,...$ The eigenfunctions can be assumed to be real. They are orthogonal, i.e.

$$\int_0^1 \psi_m\psi_n\,d\sigma = 0 \ \text{ if } \ m \neq n \ . \tag{17}$$

It is convenient to normalise them so that they satisfy

$$\int_0^1 \psi_n^2\,d\sigma = 1 \ . \tag{18}$$

The corresponding eigenvalues are real and non-negative. Clearly, owing to the no-flux boundary conditions (16), there exists a zero eigenvalue and the associated eigenfunction is equal to unity. Accordingly, the eigenvalues can be arranged in such a way that they satisfy the inequalities

$$\lambda_0 < \lambda_1 < \lambda_2 < ... < \lambda_n < \lambda_{n+1} < ... \ , \tag{19}$$

with

$$\lambda_0 = 0 \ , \ \ \psi_0(1) = 1 \ . \tag{20}$$

Then, the orthogonality constraint (17) implies that the depth mean of the eigenfunctions of order larger than or equal to unity must be zero:

$$\int_0^1 \psi_n \, d\sigma \;=\; 0 \;\text{ if }\; n \geq 1 \;. \qquad (21)$$

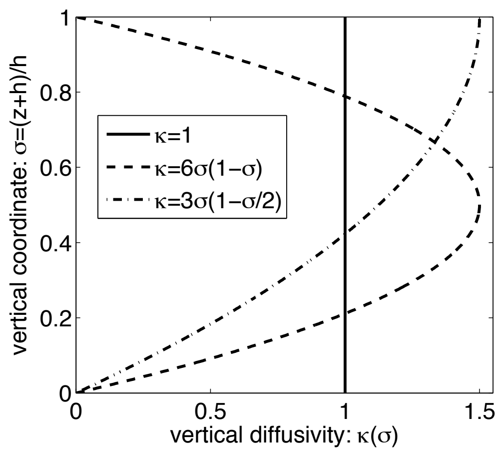

Figure 2. The profiles of the (dimensionless) vertical diffusivity, $\kappa(\sigma)$, considered in this study. In accordance with (5), the depth mean of $\kappa(\sigma)$ is equal to unity for every expression of the diffusivity.

Table 1. The eigenvalues λ_n and eigenfunctions $\psi_n(\sigma)$ for the diffusivity profiles considered in this study. The order of the mode is identified by the integer index n, with n=0,1,2,... The symbol P_n represents the n-th order Legendre polynomial.

$\kappa(\sigma)=1$	$\kappa(\sigma)=6\sigma(1-\sigma)$	$\kappa(\sigma)=3\sigma(1-\sigma/2)$
$\lambda_n = n^2\pi^2$	$\lambda_n = \frac{3}{2}n(n+1)$	$\lambda_n = 3n(2n+1)$
$\psi_0 = 1$	$\psi_0 = 1$	$\psi_0 = 1$
$\psi_n = 12\cos(n\pi\sigma)$ $(n=1,2,3,...)$	$\psi_n = \sqrt{2n+1}\,P_n(-1+2\sigma)$ $(n=1,2,3,...)$	$\psi_n = \sqrt{4n+1}\,P_n(1-\sigma)$ $(n=1,2,3,...)$

Several vertical diffusivity profiles are worth considering. The simplest of them is $\kappa = 1$, i.e. the diffusivity is constant. Then, to account for the presence of the upper and lower boundaries, the parabolic profile $\kappa = 6\sigma(1-\sigma)$ is appropriate; in the vicinity of the seabed, the diffusivity increases as a linear function of the distance to the boundary, which is consistent with the existence of the logarithmic layer. Though the bottom is generally regarded as a solid boundary, the ocean-atmosphere interface is a freely-moving boundary. To take into account the difference in the nature of the lower and upper boundaries, another diffusivity profile may also be worth studying, namely $\kappa = 3\sigma(1-\sigma/2)$ (Figure 2). The corresponding eigenvalues and eigenfunctions are listed in Table 1, while Table 2 provides, for n=1,2,3 the explicit expression of the eigenfunctions for the two parabolic diffusivity profiles. These results are illustrated in Figure 3.

Table 2. Explicit expressions of the eigenvalues λ_n and eigenfunctions $\psi_n(\sigma)$ for $n = 1,2,3$ for the parabolic diffusivity profiles considered in this study.

$\kappa(\sigma) = 6\sigma(1-\sigma)$	$\kappa(\sigma) = 3\sigma(1-\sigma/2)$
$\psi_1 = \sqrt{3}(-1+2\sigma)$	$\psi_1 = \frac{\sqrt{5}}{2}(2-6\sigma+3\sigma^2)$
$\psi_2 = \sqrt{5}(1-6\sigma+6\sigma^2)$	$\psi_2 = \frac{\sqrt{9}}{8}(8-80\sigma+180\sigma^2-140\sigma^3+35\sigma^4)$
$\psi_3 = \sqrt{7}(-1+12\sigma-30\sigma^2+20\sigma^3)$	$\psi_3 = \frac{\sqrt{13}}{16}(16-336\sigma+1680\sigma^2-3360\sigma^3+3150\sigma^4-1386\sigma^5+231\sigma^6)$

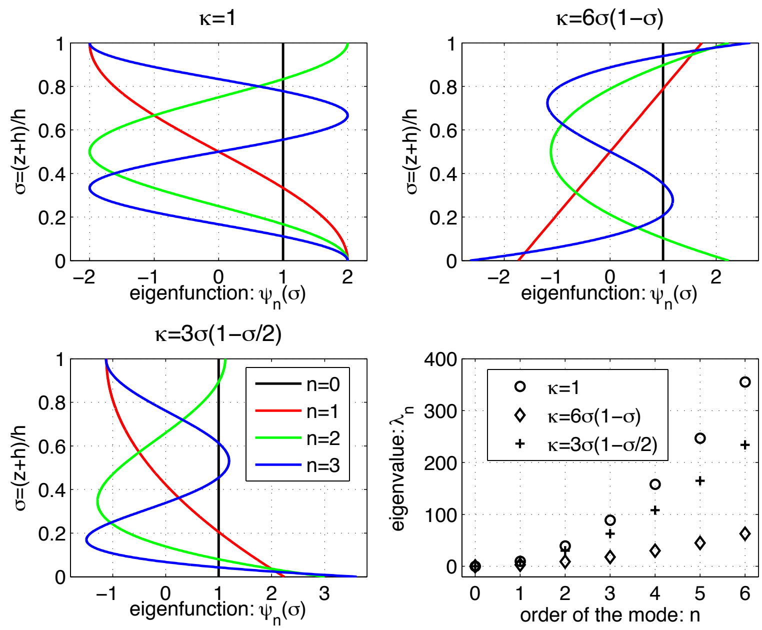

Figure 3. Graphical representation of the first eigenfunctions and eigenvalues of the vertical diffusion operator for the three vertical diffusivity profiles considered in this study.

Global mass budget

The mass of tracer injected per unit time into the domain of interest by the source is

$$Q(t) = \rho \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{1} q(t,x,y,\sigma)\, \underbrace{h\, d\sigma}_{=dz}\, dy\, dx \ . \tag{22}$$

On the other hand, the rate at which tracer mass is lost due to the first order process is $-\gamma m(t)$, because such a decay process proceeds at the same rate at any time and position. Since there is no tracer flux crossing the upper and lower boundaries of the domain, the mass of tracer present in the domain is governed by the ordinary differential equation

$$\frac{dm}{dt} = -\gamma m + Q \ , \tag{23}$$

which is to be solved under the initial condition

$$m(0) = m^0 \ . \tag{24}$$

The solution of (23)-(24) is readily seen to be

$$m(t) = m^0 e^{-\gamma t} + \int_0^t Q(t') e^{-\gamma(t-t')}\, dt' \tag{25}$$

It must now be seen that the general solution (10) satisfies (25). To do so, one must first establish the relation between the total tracer mass contained in the domain of interest and the local tracer concentration:

$$m(t) = \rho \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{1} C(t,x,y,\sigma)\, \underbrace{h\, d\sigma}_{=dz}\, dy\, dx \ . \tag{26}$$

Obviously, the initial mass obeys

$$m^0 = \rho \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{1} C^0(x,y,\sigma)\, \underbrace{h\, d\sigma}_{=dz}\, dy\, dx \ . \tag{27)}$$

Next, using the following property (which is sometimes referred to as Poisson integral)

$$\int_{-\infty}^{\infty} e^{-a\varsigma^2}\, d\varsigma = \sqrt{\frac{\pi}{a}} \ , \quad a > 0 \tag{28}$$

the integral over the domain of the Green's function (14) may be tackled, leading to the following developments

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{1} G(t,x,y,\sigma)\, d\sigma\, dy\, dx &= \overbrace{\int_{-\infty}^{\infty} (4\pi K t)^{-1/2} \exp\left[-\frac{(x-Ut)^2}{4K_h t}\right] dx}^{=1,\ \text{see (28)}} \\ &\times \underbrace{\int_{-\infty}^{\infty} (4\pi K t)^{-1/2} \exp\left[-\frac{y^2}{4K_h t}\right] dy}_{=1,\ \text{see (28)}} \times \sum_{n=0}^{\infty} \exp\left(-\frac{\bar{K}_v \lambda_n t}{h^2}\right) \psi_n(\sigma') \underbrace{\int_0^1 \psi_n(\sigma)\, d\sigma}_{\substack{=1 \text{ for n=0, see (18)} \\ =0 \text{ for n}\geq 1\text{, see (21)}}} \end{aligned} \tag{29}$$

which, since $\lambda_0 = 1$ and $\psi_0 = 1$, simplifies to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{1} G(t,x,y,\sigma)\, d\sigma\, dy\, dx \;=\; 1 \;. \tag{30}$$

This result should come as no surprise, since the Green's function (as defined herein) represents the "concentration" associated with an abrupt, unit release of a passive tracer. Finally, combining (10), (26), (27) and (30) yields

$$\begin{aligned}
m(t) &= \rho h \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{1} C(t,x,y,\sigma)\, d\sigma\, dy\, dx \\
&= \rho h \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{1} C^0(x',y',\sigma')\, e^{-\gamma t} \overbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{1} G(t,x-x',y-y',\sigma,\sigma')\, d\sigma\, dy\, dx}^{=1,\text{ see (30)}}\, d\sigma'\, dy'\, dx' \\
&\quad + \rho h \int_{0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{1} q(t',x',y',\sigma')\, e^{-\gamma(t-t')} \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{1} G(t,x-x',y-y',\sigma,\sigma')\, d\sigma\, dy\, dx}_{=1,\text{ see (30)}}\, d\sigma'\, dy'\, dx'\, dt' \\
&= \rho h \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{1} C^0(x',y',\sigma')\, e^{-\gamma t}\; d\sigma'\, dy'\, dx' \;+\; \rho h \int_{0}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{1} q(t',x',y',\sigma')\, e^{-\gamma(t-t')}\; d\sigma'\, dy'\, dx'\, dt' \\
&= e^{-\gamma t}\, \underbrace{\rho h \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{1} C^0(x',y',\sigma') d\sigma'\, dy'\, dx'}_{=m^0,\text{ see (27)}} \;+\; \int_{0}^{t} e^{-\gamma(t-t')} \underbrace{\rho h \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{1} q(t',x',y',\sigma')\, d\sigma'\, dy'\, dx'}_{=Q(t'),\text{ see (22)}}\, dt' \\
&= m^0\, e^{-\gamma t} \;+\; \int_{0}^{t} Q(t') e^{-\gamma(t-t')}\, dt'
\end{aligned} \tag{31}$$

As expected, this expression is equivalent to (25). QED.

Depth-averaged concentration

Let an overbar denote the depth mean of a variable, i.e.

$$\bar{\xi}(t,x,y) \;\equiv\; \int_{0}^{1} \xi(t,x,y,\sigma)\, d\sigma \;. \tag{32}$$

Then, using (14), (20)-(21), the depth-averaged Green's function is readily seen to be

$$\bar{G}(t,x,y) \;=\; \frac{1}{4\pi K_h\, t} \exp\left[-\frac{(x-Ut)^2+y^2}{4K_h\, t}\right] \;. \tag{33}$$

Next, taking the depth mean of the concentration (10) and using (33) yields

$$\begin{aligned}
\bar{C}(t,x,y) &\equiv \int_{0}^{1} C(t,x,y,\sigma)\, d\sigma \;= \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{1} C^0(x',y',\sigma')\, e^{-\gamma t} \overbrace{\int_{0}^{1} G(t,x-x',y-y',\sigma,\sigma')\, d\sigma}^{=\bar{G}(t,x-x',y-y')}\, d\sigma'\, dy'\, dx'
\end{aligned}$$

$$+\int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^1 q(t',x',y',\sigma')\, e^{-\gamma(t-t')} \underbrace{\int_0^1 G(t-t',x-x',y-y',\sigma,\sigma')\, d\sigma}_{=\bar{G}(t,x-x',y-y')}\, d\sigma'\, dy'\, dx'\, dt'$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overbrace{\int_0^1 C^0(x',y',\sigma')\, d\sigma'}^{=\bar{C}^0(x',y')}\, e^{-\gamma t}\, \bar{G}(t,x-x',y-y')\, dy'\, dx'$$

$$+\int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underbrace{\int_0^1 q(t',x',y',\sigma')\, d\sigma'}_{=\bar{q}(t',x',y')}\, e^{-\gamma(t-t')}\, \bar{G}(t-t',x-x',y-y')\, dy'\, dx'\, dt' \tag{34}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{C}^0(x',y')\, e^{-\gamma t}\, \bar{G}(t,x-x',y-y')\, dy'\, dx' \; +$$

$$+\int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{q}(t',x',y') e^{-\gamma(t-t')}\, \bar{G}(t-t',x-x',y-y')\, dy'\, dx'\, dt'$$

Combining (33) and (34) leads to the general expression of the depth-averaged concentration

$$\bar{C}(t,x,y) = \frac{e^{-\gamma t}}{4\pi K_h t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{C}^0(x',y') \exp\left[-\frac{(x-x'-Ut)^2+(y-y')^2}{4K_h t}\right] dy'\, dx' \; +$$
$$+\int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{q}(t',x',y')\, \frac{e^{-\gamma(t-t')}}{4\pi K_h (t-t')} \exp\left[-\frac{(x-x'-Ut+Ut')^2+(y-y')^2}{4K_h (t-t')}\right] dy'\, dx'\, dt' \tag{35}$$

Clearly, the depth-averaged concentration is independent of the vertical dependency of the initial concentration and the source functions. Only their depth means impact the depth-averaged concentration. This result, which is not unexpected, is essentially due to the horizontal velocity being independent of the vertical coordinate in the flow considered herein.

The solution (35) may be seen to satisfy the equation governing the evolution of the depth-averaged concentration, which is obtained by integrating (7) over the height of the water column and taking into account the impermeability conditions (9), i.e.

$$\frac{\partial \bar{C}}{\partial t} + U \frac{\partial \bar{C}}{\partial x} = \bar{q} - \gamma \bar{C} + K_h \frac{\partial^2 \bar{C}}{\partial x^2} + K_h \frac{\partial^2 \bar{C}}{\partial y^2} \; . \tag{36}$$

Simple solutions

Though the solution (10) appears to be rather intricate, the response to elementary forcings may be quite simple. For instance, if there is no tracer in the domain at the initial instant ($C^0 = 0$) and a mass M of tracer is injected abruptly at point (x_i, y_i, σ_i) at time t=0, then the corresponding source function is

$$q(t,x,y,\sigma) = \frac{M}{\rho h}\, \delta(x-x_i)\, \delta(y-y_i)\, \delta(\sigma-\sigma_i)\, \delta(t-0) \; . \tag{37}$$

Substituting this into the general expression (10) yields

$$C(t,x,y,\sigma) = \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^1 \frac{M}{\rho h}\, \delta(x'-x_i)\, \delta(y'-y_i)\, \delta(\sigma'-\sigma_i)\, \delta(t'-0) \times e^{-\gamma(t-t')}\, G(t-t',x-x',y-y',\sigma,\sigma')\, d\sigma'\, dy'\, dx'\, dt' \tag{38}$$

$$= \frac{M}{\rho h}\, e^{-\gamma t}\, G(t,x-x_i,y-y_i,\sigma,\sigma_i)$$

Then, combining (14) and (38) leads to an analytical solution whose value is easy to calculate at any time and location (since no integral is to be evaluated numerically):

$$C(t,x,y,\sigma) = \frac{M}{\rho h} \frac{e^{-\gamma t}}{4\pi K_h t} \exp\left[-\frac{(x-x_i-Ut)^2+(y-y_i)^2}{4K_h t}\right] \times \sum_{n=0}^{\infty} \exp\left(-\frac{\bar{K}_v \lambda_n t}{h^2}\right) \psi_n(\sigma_i)\, \psi_n(\sigma) \tag{39}$$

The corresponding depth mean concentration is readily seen to be

$$\bar{C}(t,x,y) = \frac{M}{\rho h} \frac{e^{-\gamma t}}{4\pi K_h t} \exp\left[-\frac{(x-x_i-Ut)^2+(y-y_i)^2}{4K_h t}\right] \tag{40}$$

The same solution is obtained if the source function is assumed to be zero ($q=0$) and the initial condition tracer distribution is as follows

$$C^0(t,x,y,\sigma) = \frac{M}{\rho h}\, \delta(x-x_i)\, \delta(y-y_i)\, \delta(\sigma-\sigma_i) \ . \tag{41}$$

This equivalence is due to the fact that abruptly injecting a mass M of tracer at time t=0 at location (x_i,y_i,σ_i) is equivalent to assuming that at the initial instant a mass M of tracer is concentrated at the point (x_i,y_i,σ_i). A thorough demonstration thereof is readily achieved.

Now assume that a point source located at (x_i,y_i,σ_i) injects tracer into the flow at constant rate Q, i.e. the mass of tracer released during the time interval $[t,t+\delta t]$ is equal to $Q\,\delta t$. The corresponding source function is

$$q(t,x,y,\sigma) = \frac{Q}{\rho h}\, \delta(x-x_i)\, \delta(y-y_i)\, \delta(\sigma-\sigma_i) \ . \tag{42}$$

If the initial concentration is zero, the tracer concentration is

$$C(t,x,y,\sigma) = \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^1 \frac{Q}{\rho h}\, \delta(x'-x_i)\, \delta(y'-y_i)\, \delta(\sigma'-\sigma_i) \times e^{-\gamma(t-t')}\, G(t-t',x-x',y-y',\sigma,\sigma')\, d\sigma'\, dy'\, dx'\, dt' \tag{43}$$

$$= \frac{Q}{\rho h} \int_0^t e^{-\gamma(t-t')}\, G(t-t',x-x_i,y-y_i,\sigma,\sigma_i)\, dt'$$

A steady state is reached in the limit $t \to \infty$:

$$C(\infty,x,y,\sigma) = \frac{Q}{\rho h} \int_0^{\infty} e^{-\gamma\xi}\, G(\xi,x-x_i,y-y_i,\sigma,\sigma_i)\, d\xi$$

$$= \frac{Q}{\rho h} \int_0^\infty \frac{\exp\left[-\gamma\xi - \frac{(x-x_i-U\xi)^2-(y-y_i)^2}{4K_h\,\xi}\right]}{4\pi K_h\,\xi} \sum_{n=0}^{\infty} \exp\left(-\frac{\bar{K}_v\lambda_n\xi}{h^2}\right) \psi_n(\sigma')\,\psi_n(\sigma)\;d\xi \;. \qquad (44)$$

Next, using the following property

$$\int_0^\infty \frac{e^{-a/\varsigma - b\varsigma}}{\varsigma}\, d\varsigma \;=\; 2K_0\left(\sqrt{4ab}\right)\,, \quad a,b>0\;, \qquad (45)$$

where K_0 denotes the modified Bessel function of the second kind of order zero, the integral (44) may be evaluated, yielding after some calculations

$$C(\infty,x,y,\sigma) \;=\; \frac{Q}{2\pi\rho h K_h} \exp\left(\frac{Ux}{2K_h}\right) \sum_{n=0}^{\infty} K_0\left(\mu_n\sqrt{(x-x_i)^2+(y-y_i)^2}\right) \psi_n(\sigma_i)\,\psi_n(\sigma) \qquad (46)$$

with

$$\mu_n \;=\; \sqrt{\frac{\gamma}{K_h} + \frac{\bar{K}_v\,\lambda_n}{h^2 K_h} + \frac{U^2}{4K_h^2}}\;. \qquad (47)$$

The associated depth-averaged concentration is readily seen to be

$$\bar{C}(\infty,x,y) \;=\; \frac{Q}{2\pi\rho h K_h} \exp\left(\frac{Ux}{2K_h}\right) K_0\left(\mu_0\sqrt{(x-x_i)^2+(y-y_i)^2}\right) \qquad (48)$$
